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A Dialectical Approach to the Formation of Mathematical Abstractions

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This paper is structured in two sections. The first examines views of mathematical abstraction in two broad categories: empiricist and dialectical accounts. It documents the difficulties involved in and explores the potentialities of both accounts. Then it outlines a recent model which takes a dialectical materialist approach to abstraction in context. This model constitutes the basis of the second section where we describe an empirical study designed to investigate mathematical abstraction in socially rich (e.g., peer-interacted and tutor-assisted) environments. We then present data on two students working with the help of a tutor on tasks concerned with graphs of absolute value functions. On the basis of these data, we discuss four particular themes which are relevant to the purpose of this special issue and are important in the discussion of mathematical abstraction: human and artefact mediation, tutor interventions in assisting the formation of mathematical abstractions, implications of a dialectical view on student development, and the things that are abstracted.

Abstraction is an important construct for mathematics educators. Empiricist accounts of abstraction, which associate abstractions with generalisations stemming from the recognition of common features amongst a set of particular instances, dominated Western mathematics education literature in the last century but have been challenged in recent years by ideas inspired by the dialectical materialist and situated accounts of human cognition.

This paper outlines a dialectical materialist[1] approach to abstraction and broadly addresses the following questions:

- How are abstractions mediated and what kind of mediation are we talking about?
- What is the role of tutor interventions in guiding students?
- What are the implications of a dialectic view of abstraction on student development?
- What kind of things are abstracted?

In this paper we first attend to empiricist accounts of abstraction and discuss difficulties involved in these accounts. We do this because these accounts are so pervasive. We then focus on dialectical materialist accounts of abstraction, paying particular attention to the works of Davydov (1972/1990) and Hershkowitz, Schwarz, and Dreyfus (2001), two accounts that have had a great impact on our thoughts on abstraction. We then describe and provide data from empirical research designed to investigate student abstraction within the framework of the model of Hershkowitz et al. Finally, we return to the four questions above.

1 Dialectical materialism is used in the sense explained in Chapter 7 of Davydov (1972/1990): "An individual person's thought is the functioning of historically developed forms of society's activity which have been conferred on him" (p. 232).

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Empiricist Accounts of Abstraction

Before going into the detailed consideration of the issues, we feel it necessary to note that the term *abstraction* refers to a process leading to a product. The debate between empiricist and dialectical accounts is often centred on the nature of the abstraction process rather than the product, and these accounts lay the emphasis on the investigation of process aspects of abstraction. Although our considerations below focus on issues related to process aspects of abstraction we believe that the end product of an abstraction process is of great importance. We attend to these matters in the discussion section of this paper in the light of our data in reflecting on the question: What kind of things are abstracted?

In the Western world, abstraction is often linked with empiricist philosophy and extracting ideas from their material origins, as expressed by Locke (1689/1964, p. 264): "Ideas become general by separating from them the circumstances of time and place." Such views had a great impact on mathematics education thought in the 20th century. Dreyfus (1991), for example, writes of generalisation — a component part of abstraction: "To generalize is to derive or induce from particulars, to identify commonalities" (p. 35). Dienes (1963) describes abstraction as

> ... the extraction of what is common to a number of different situations. It is just another word for the formation of a class, the end-point being the realisation of the attribute or attributes which make elements eligible or not for membership of the class. (p. 57)

Piaget (2001) distinguishes three forms of abstraction: empirical abstraction that is concerned with the features of the objects available to the senses (e.g., the weight and colour of a pebble); pseudo-empirical abstraction concerned with the actions on the objects (e.g., counting the pebbles); and reflective abstraction concerned with the interrelationships amongst the actions on objects (e.g., discovering commutativity through the consideration of counting actions).

Empiricist accounts of abstraction have three essential features: (a) Abstraction is derived from the recognition of commonalities across a set of particular instances; (b) abstraction is a process of decontextualisation (i.e., separation of the circumstances of time and place); and (c) abstraction is a development ascending from the concrete to the abstract. We argue that these three features are problematic and briefly attend to the problems involved in (a) and (b). We consider problems involved in (c) in the next section on dialectical accounts of abstraction.

Regarding (a), viewing abstractions as stemming from the recognition of commonalties amongst particular instances is problematic with regard to the epistemological primacy of particular instances over abstractions — how can one recognise particulars in one set as instances of an abstraction unless this person has, at least, a rudimentary understanding of that abstraction? This view, as van Oers (2001, p. 283) notes, commits "the fault of a *petitio principii*, for what comes out of the act of abstraction ... is already presumed." As Ohlsson and Lehtinen (1997, p. 41) suggest, "people experience particulars as similar precisely to the

extent that, and because, those particulars are recognised as instances of the same abstraction" and hence for the recognition of "an object as an instance of an abstraction, the knower must already possess that abstraction."

With regard to the second feature, in empiricist philosophy the end result of an abstraction process is viewed as the production of an *entity* which is decontextualised (e.g., Locke on separating ideas "from the circumstances of time and space"). Noss and Hoyles (1996) see the difficulty involved in the idea of abstraction as a process of decontextualisation and for them this is a problem of meaning:

> Where can meaning reside in a decontextualised world? If meanings reside only within the world of real objects, then mathematical abstraction involves mapping meaning from one world to another, meaningless, world — certainly no simple task even for those with the capacity to do it. If meaning has to be generated from within mathematical discourse without recourse to real referents, is this not inevitably impossible for most learners? (p. 21)

These authors make a compelling case and show that the specificities of the context (such as the tools and artefacts that the students have at their disposal, the students' previous actions and the presentation of the tasks) make differences in students' formation of abstractions. We are aware that the term *context* is a problematic one (see Cole, 1996), yet our own position is that abstractions arise and are applied in contexts. We see insuperable problems with a view of abstractions as ideas unrestricted with regard to time and space. We are convinced that the formation of abstractions and the use of formed abstractions involve and depend, for example, on tools and artefacts available in a learning situation, the social and personal histories of the learners, and the cultural and historical milieu that surrounds them (classmates, teachers, others), the influence of institutions and value judgements (Ozmantar & Monaghan, 2005). We hope that the contextualised nature of abstraction will become clearer in the discussion section at the end of this paper.

Dialectical Accounts of Abstraction

The empiricist view of abstraction may be regarded as an ascent from the concrete to the abstract. The concrete is associated with physical knowledge — knowledge based on experience — and the abstract is associated with logical and mental structures such as mathematics (Piaget, 1970; van Oers, 2001). Such an association privileges the abstract over the concrete, as recognised by Noss and Hoyles (1996, p. 45):

> A standard description of the difference between thinking in lived-in experience and mathematical thinking is that the former is concrete, the latter abstract. There is no doubt which way the hierarchy sits: the history of Western thought has privileged the latter at the expense of the former. Abstract is general, decontextualised, intellectually demanding; concrete is particular, context-bound, intellectually trivial.

However, abstraction can also be conceived of not so much as a step upwards in an ascending process but rather as an "intertwining of theories, experiences and previously disconnected fragments of knowledge" (Noss & Hoyles, 1996, p. 44). In a similar vein, van Oers (2001), drawing on Marxist theory, argues against a division of the concrete and abstract on the basis of perceptual-material and mental-conceptual world:

> The split between the concrete and the abstract actually creates a misleading divorce between the perceptual-material and the mental-conceptual world. Abstraction can never produce meaningful insights in the concrete world, unless there is some inner relationship between the concrete and the abstract (p. 287).

For van Oers, the concrete and the abstract are dialectically connected in the activities of human beings whose actions are intrinsically related to the activity setting which represents a multi-faceted yet organised whole. This, following Marxist terminology, is the "unity of the diversity" and is concrete. Abstraction is a process of making sense of such concrete situations by discovering new meanings in order to establish interconnections amongst the different elements of the whole.

A second aspect of empiricist view of the concrete and the abstract is that the abstract may be viewed as the *essence* that a number of particulars have in common (see Brook, 1997); that is, the abstract is associated with the essence and the concrete refers to the particulars from which this essence is produced. This view assumes a mechanism by which "the essence can be extracted from an object, either by stripping off accidental, irrelevant features or by directly focusing on the essence (prototype, scheme)" (van Oers, 2001, p. 283). But how is one guided in deciding what is relevant and what is irrelevant, and hence determining the essence? Several authors point to certain innate human characteristics that lead to discovering the essence. Bartlett (1932), for example, writes about the abstraction of schemas and argues that in the course of abstraction humans are engaged in a kind of involuntary analysis which focuses on certain elements of a whole while ignoring others. To Bartlett, this is the work of the operation of persistent interests in humans. Similarly, Keil (1989) proposes that humans have a peculiar innate tendency to embrace causal beliefs.

However, such theories do not take the analysis further as they postulate a mysterious mechanism which works to discover the essence. Davydov (1972/1990, pp. 277-278) addresses this problem as follows:

> The central or the essential must be detached from the structure of random abstractions, and, furthermore, in thought one must keep to the essence of the matter rather than to accessory mediating qualities that exist everywhere in a complex whole. But where can we obtain a criterion for 'essentialness' — and then how can one be guided by it in choosing initial abstractions, for example? In themselves they do not have this criterion. Among them it is impossible to delineate the initial and the subsequent, the central and the noncentral, in an unambiguous way. Traditional formal logic does not formulate any rules on this score.

Davydov presents a Marxist formulation: The concrete is correlated with the abstract, and development during abstraction is not an ascent from the concrete but a dialectical, two-way relationship between the concrete and the abstract. In Davydov's view, the concrete refers to:

> ... some developed whole, interconnection, unity of different aspects — it is the synonym for the determining role of the whole with respect to its parts, features, and aspects. "The abstract" usually has several characteristics — it is something simple, devoid of differences, fragmentary, and undeveloped. (1990, p. 283)

In his view, the discovery of the essence is the result of an abstraction process which ultimately needs to ascend back to the concrete:

> To know essence means to find the universal as a base, as a single source for a variety of phenomena, and then to show how this universal determines the emergence and interconnection of phenomena — that is, the existence of concreteness. (p. 289)

To do so, one needs to engage in analysis and synthesis. Analysis often takes place in the initial stages of an abstraction and is necessary to establish initial abstractions — that is, undeveloped and yet to be particularised relationships among the different elements of the whole. During the analysis, external (observable) features of reality are connected by empirical thought (observing similarities and differences, designating contradictions). However, determining the essence also requires synthesis on the basis of the findings from the analysis by employing theoretical thought which reproduces "the universal forms of things, their measures and their laws" (Davydov, p. 249). Theoretical thought establishes the essential relationships which are not directly available to the senses. The end product of the abstraction process is the development of a final form which is consistent and highly structured (i.e., concreteness).

For Davydov, the source and basis of an abstraction is practical human activity where individuals draw on the features and the potentialities of the objects (tools, concepts). He also points to the importance of the product of historically established concreteness: "Outside of the mind of the knowing person, there exist individual, particular things and phenomena that function as products and features in the development of a certain concreteness" (1990, p. 285). Social interaction and other individuals play a central role in Davydov's ascent to the concrete through analysis and synthesis, but in Davydov (1972/1990) he says virtually nothing on this, especially with regard to practice in educational settings.

Davydov's method of ascent is echoed in the writings of van Oers (2001), who argues that abstraction takes place in socio-culturally constructed activities from which new actions emerge. For him, abstraction is the process of contextualising an experience from a certain point of view (relation, metaphor, image) and the selection of this viewpoint is what guides individuals towards the discovery of the essence. Viewpoints are not pre-given entities but are constructed in communicative interpretative processes of social interaction

involving goal-driven, tool-mediated human actions. He elaborates, on the basis of empirical data, student's selection of certain viewpoints and argues that understanding develops in a discursive process by which meaning is negotiated accordingly. He argues that students should be given a perspective on where they are going by having a role in the contextualisation process itself. Hence students need to be directed in the given situation:

> It is the expert (teacher) who discursively focuses pupils on particular and increasingly "isolated" aspects of the situation and helps the children in the construction of new mental objects ("abstractions"), that provide the means for seeing various things as related and thus to ascend from the abstract to the concrete. (pp. 300-301)

Hershkowitz, Schwarz, and Dreyfus (2001) developed the RBC (recognising — building-with — constructing) model of abstraction, based on activity theory (Leont'ev, 1981), which embraces Davydov's method of ascent. In the RBC model, particular importance is attached to theoretical thought (in the sense of Davydov) in the formation of abstractions, although they note that empirical thought may also be employed during this process. Davydov's dialectic approach is also evident in the RBC model, in which the abstraction process is seen as starting from an initial, unrefined, first form and proceeding to the production of a highly consistent final structure through reorganisation of available structures and the establishment of new connections amongst them.

More precisely, the RBC model treats abstraction as an activity of vertically reorganising previously constructed mathematical knowledge into new knowledge structures. *Vertical reorganisation* refers to the integration of mathematical elements and their development into more complex knowledge structures. The model, like other dialectical accounts, is sensitive to contextual parameters and posits that development during abstraction occurs through mediational means and social interactions. In their view, the abstraction process commences with a need for a new structure and consists of three nested epistemic actions: recognising, building-with, and constructing. *Recognising* refers to the realisation of a mathematical structure/element that is already familiar to the learner as a result of earlier abstractions. Here, the term recognition is used in a different sense to empiricists' use of the term: In the RBC model, recognising involves empirical thought but empirical thought alone is insufficient to form abstractions, which require theoretical thought; whereas empiricists focus on recognition as a means of producing the abstractions. The second epistemic action, *building-with*, refers to the use of available mathematical elements in achieving a goal such as solving a problem or giving an explanation. Finally, *constructing* actions refer to the assembly of mathematical structures in producing a novel one.

These epistemic actions are goal-driven and tool-mediated actions taking place in a social and historical context, are part of a greater activity, and can only be meaningful within the context of this activity. It is through these actions that new structures are constructed. The model also assumes the importance of

consolidating newly constructed structures and an abstraction progresses through three stages: the need for a structure, the construction of a new structure through nested epistemic actions, and consolidation of the new structures.

This, we believe, is an important model as it allows one to be precise about microgenetic development during the abstraction process and it is open to empirical investigation. The model has been the basis for a number of research studies (Bikner-Ahsbahs, 2004; Dreyfus & Tsamir, 2004; Stehlikova, 2003; Tsamir & Dreyfus, 2002; Williams, 2003, 2004; Wood & McNeal, 2003; Ozmantar & Roper, 2004). Ozmantar (2005) is an empirical study which investigates the validity of this model by considering epistemological and sociocultural principles, epistemic actions, and the genesis of abstractions. Findings from this study refined aspects of the RBC model — in particular, consolidation. In Monaghan and Ozmantar (2006), we detailed the nature of the consolidation process and argued that newly constructed structures are fragile entities and in need of consolidation. Our observations led us to conclude that an abstraction, as a product, is a consolidated construction that can be used in further abstractions. The following section outlines our study.

The Study, the Task and the Protocol Data

The study investigated the validity of the RBC model of abstraction with regard to aspects of human interactions, including scaffolding (Wood, Bruner, & Ross, 1976) and peer interactions. Our focus on scaffolding developed from a realisation that tutor intervention designed to assist students towards the achievement of abstraction, is an important, but largely uncommented on, aspect of the RBC model. Our intention to examine the model in relation to peer interaction was motivated by our desire to understand the formation of abstractions in socially rich environments.

To realise our aim, we collected data from Turkish students working with and without the assistance of a knowledgeable agent (the first author as interviewer) on tasks related to the absolute value of linear functions. The notions of absolute value and the graphing of absolute value functions form part of the Turkish National Curriculum and are tested in the high-stakes university entrance examination. According to the Curriculum Standards, Turkish students are expected to comprehend the application of absolute value in the algebraic domain in Grade 9 and to be able to graph absolute value functions in Grade 11. So we chose to focus on the understanding of linear absolute value functions among Grade 10 students, 16-18 years old, who had the mathematical prerequisites to tackle this topic but had not studied it at school. The focus on the linear absolute value functions was opportunist: It is a topic that is intellectually demanding but achievable by the target students and it is an area where consecutive tasks can be developed to investigate the formation of abstractions and the use of these abstractions in forming further abstractions.

We designed a screening test to identify students who had the prerequisite knowledge structures necessary to carry out the task but were not familiar with the task content. One hundred and thirty-four Grade 10 students took this test

and 20 were found appropriate for the study. In collecting data from the students, the study employed a multiple case study strategy with the purpose of producing literal and theoretical replications to supply corroboratory or disconfirming evidence for the observed trends from cross examination of the cases (Yin, 1998). Six students worked individually and 14 in pairs; 3 of the individuals and 4 of the pairs worked with the interviewer scaffolding their work and others worked on their own.

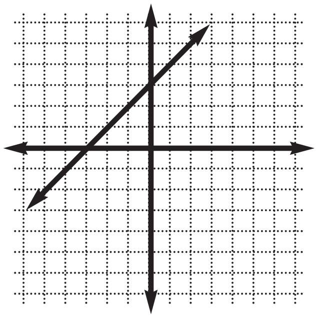
The students worked four consecutive days on four tasks without time limitations. The first two tasks focused on sketching the graphs of $y = |f(x)|$ and $y = f(|x|)$ using given graphs of $y = f(x)$. The third task was designed to consolidate the constructions made in the initial tasks. We included the consolidation task to enable students to attain flexible use of the constructions achieved in the first two tasks (Monaghan & Ozmantar, 2006), which were necessary to achieve the target abstraction of the final task. The fourth task, shown in Figure 1, focused on sketching the graphs of $y = |f(|x|)|$.

The interviewer scaffolded students' work by providing purposeful assistance — by asking them to clarify or explain their actions, giving direction if needed; intervening if necessary, prompting them to explain what they were doing and why they were doing it; encouraging the students to reflect on the problems they were solving and to analyse the contribution of their actions to their solutions. Note that scaffolding is not the main focus of this paper; we provide this description only to give the reader an appreciation of the interview situation.

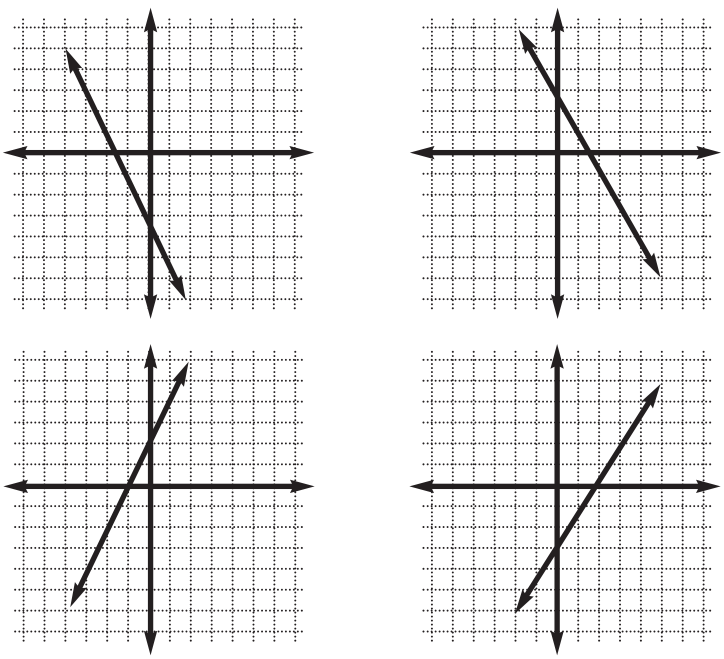
Students' verbalisations during the task performance were audio-recorded, and written responses, computations, and graphs were retained. Protocols were then compiled from a simultaneous examination of the students' written responses and transcriptions of verbal utterances. The protocols were in Turkish and later translated into English. In translation, particular attention was paid to reflect the students' original intent as clearly as possible.

While analysing the verbal data, we used the RBC model as a framework and also focused on epistemic actions to better appreciate the development of students' mathematical abstraction. Repeated readings of the verbal data led us to make certain observations (e.g., human and artefact mediation), and adapting a multiple case study strategy enabled us to verify our observations by a cross examination of the cases. To guide the reader through the development of our observations and for the establishment of our interpretations we will provide, in the discussion section, a detailed description of the excerpts. However, we cannot provide a precise analysis of the protocols in terms of epistemic actions, as this would enlarge the paper beyond the acceptable limits; readers are referred to Ozmantar (2005) for such an analysis of the excerpts and also for the further details of our analysis procedures.

1. A function of f is defined on the set of real numbers as $f(x) = x - 4$. Draw the graph of $y = |(|x| - 4)|$ and comment on any patterns or symmetries.
2. Do you see any relationship between the graph of $f(x) = x - 4$ and the graph of $x = |(|x| - 4)|$? Explain your answer.
3. The graph of $f(x) = x + 3$ is given below. Can you obtain the graph of $y = |(|x| + 3)|$ from the graph of $f(x)$? Explain your answer.



4. There are four different graphs of $f(x)$ given below. Find the graphs of $|f(|x|)|$ by making use of the graphs of $f(x)$.



5. How would you explain to your friend how to draw the graph of $|f(|x|)|$ by using the graph of $f(x)$? Demonstrate that your explanation is correct by using the above-given graphs.

Figure 1. Task 4.

Protocol Data

We focus here on the protocol of a pair of 17-year-old girls, called H&S, working on Task 4. We have divided H&S's protocol into four episodes which, in our opinion, reflect distinct stages in the students' knowledge development.

In the following excerpts, "H" and "S" refer to the two girls and "I" to the interviewer. We occasionally insert words in square brackets to assist the reader; we also sometimes omit parts of the participants' utterances, indicated as (...). To save space, we refer to the graphs in question simply as $|f(x)|$, $f(|x|)$, and $|f(|x|)|$.

In Questions 1 and 2 of Task 4 (see Figure 1), H&S accurately sketched the graphs of $|f(|x|)|$ and $f(x)$ (see Figures 2-A & 2-B) by substitution of different values of x into the given equations to find the corresponding values of y, plotting points, and then joining them to obtain the target graphs. Afterward, they commented on the symmetry of their graph and the associated reflections. They then moved on to Question 3 of Task 4 and correctly sketched the graph, also by substitution (Figure 2-C). They were, however, confused by the different shapes of the two $|f(|x|)|$ graphs and experienced difficulties. Episode 1 shows their discussion.

Episode 1

[133] H: Look I think the first part [of $f(x)$ at $x > 0$; see Figure 2-C] always remains the same... . Oh, does it?

[134] S: Yes.

[135] H: But in the first question there is something else... I mean a line segment [see Figure 2-A].

[136] S: This graph is also symmetric in the y-axis. But I don't know how it helps us!

[137] H: We know that the part of $f(x)$ above the x-axis remains the same, right?

[138] S: Yes... it remains the same and also because y never takes negative values, they are taken symmetrically in the x-axis.

[139] H: But wait! We said this part [of $f(x)$ over the x-axis] didn't change, but in this graph [see Figure 2-C], it doesn't obey this rule... The part of $f(x)$ in the first quadrant remains the same.

[140] S: Yeah I know, there was a line segment in the first graph [see Figure 2-A].

[141] H: I don't think we can ever understand how to use *fx*) to draw the graph of $|f(|x|)|$.

[142] S: The first graph [Figure 2-A] was something like W-shaped... but this graph is V-shaped [Figure 2-C].

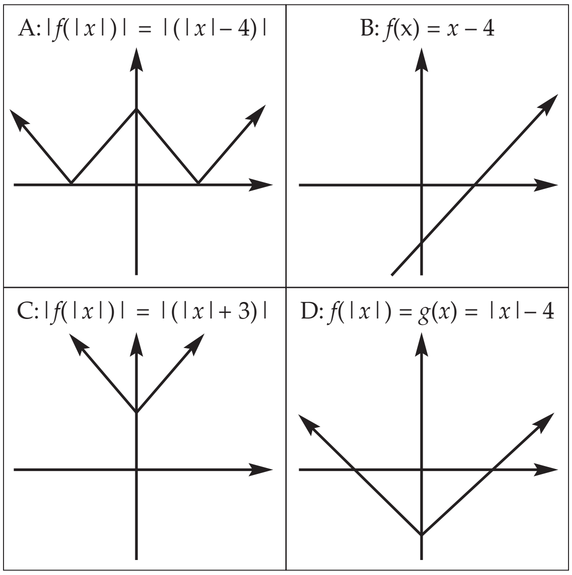


Figure 2. The graphs H&S sketched.

[143] H: They are totally different! How can we speak in a general way? Even this question made things worse rather than helping us.

[144] S: I think we better stick to substituting; we can answer the next questions by substituting.

[145] H: Yeah, after all it is definitely working to draw $|f(|x|)|$.

[146] I: Let's return to the first question once again and talk about it together, OK?

[147] H: I knew you would suggest that.

[148] S: We are useless!

Although H&S accurately sketched the target graphs, they were unable to reconcile the two representations. They were clearly having difficulties in making sense of these graphs, became frustrated, and eventually stated their intention to give up developing a new method when substitution was working well for them. They primarily focused on the visual characteristics of the graphs (line segments, rays, symmetries) but appeared unaware of the connection between their earlier work on the graphs of $|f(x)|$ and $f(|x|)$ and the target graph of this task, $|f(|x|)|$. Having realised H&S's frustration, the interviewer felt that it was time to intervene and suggested returning to the first question and working on the task together. He then (lines 150-164, not shown) drew H&S's attention to H&S's constructions of $|f(x)|$ and $f(|x|)$ in the first two tasks and suggested that they keep these in mind in this task. We resume the protocol at line 165.

Episode 2

[165] I: OK, if you pay a closer attention to the equation... I mean look at the expression itself, [$|f(|x|)|$]. It is a combination of these two, [$|f(x)|$ and f($|x|$)]. Do you see that?

[166] H: Yes, that's right. We already mentioned about this at the beginning...

[167] S: Yeah, this [$|f(|x|)|$] is a combination of $f(|x|)$ and $|f(x)|$... (...)

[168] I: OK, let's think about it [$|f(|x|)|$] and consider what you know. How can we use our knowledge to obtain this graph?

[169] S: I think it makes sense... (...)

[170] H: Yeah, it makes sense now... look, if $|f(|x|)|$ is a combination of $f(|x|)$ and $|f(x)|$, can we think about it like a computation with parentheses?

[171] I: Computation with parentheses?

[172] H: I mean, for example, when we are doing computations with some parentheses like... let's say for example, (7 – (4 + 2)), then we follow a certain order...

[173] S: Right, I understood what you mean... we need to first deal with the parenthesis inside of the expression, is that what you mean?

[174] H: Yeah, I think it is somehow similar in here, I can sense it but I am unable to clarify...

[175] S: I know what you mean but how could we determine the parenthesis in here?

[176] I: You both made an excellent point. OK, let's think about it [$|f(|x|)|$] together! In the expression of $|f(|x|)|$, can we think about the absolute value sign at the very outside of the whole expression as a larger parenthesis, which includes another one just inside?

Following the interviewer's prompt, H suggested an analogy with arithmetic (computational order and parentheses). S shared this recognition, which provided them with a starting point to build a new method. They were, however, uncertain as to how to "determine the parentheses". The interviewer intervened and explained how computational priority might work with absolute value signs. This provoked an immediate "aha!" from H at the start of the next episode:

Episode 3

[177] H: Aha, I got it... I know what we will do.

[178] I: Could you please tell us?

[179] H: We can consider $f(|x|)$ as if it was the smaller parenthesis!

[180] I: Smaller parenthesis?

[181] H: I mean it should be the first thing that we need to deal with.

[182] S: Yeah, I agree... I think we should begin with the graph of $f(|x|)$ and first draw it.

[183] H: But what next?

[184] S: Then we can use the absolute value at the outside... in the similar way to doing computations.

[185] H: But we will be drawing graphs! Can we really do this?

[186] S: I am not too sure if we can... but it sounds plausible...

[187] I: What you are doing here is not computation of course... but you are making an analogy, I mean you are making some certain logical assumptions based on your earlier experiences... and I see no problem with that... let's draw the graph by considering what we've just talked about and then decide if it will work or not, huh?

H&S collaboratively built a strategy (with H initiating and then S leading) about how to use the structures of $f(|x|)$ and $|f(x)|$ in order to sketch the graph of $|f(|x|)|$. H&S were once again uncertain if their approach was plausible. To handle this uncertainty and guide H&S, the interviewer intervened and suggested drawing the graph.

Episode 4

[188] H: What are we doing now?

[189] S: We will draw first the graph of $f(|x|)$.

[190] H: OK let's draw the graph now... [Figure 2-D]

[191] I: All right, you drew the graph of $f(|x|)$. But this is not what we are required to find, is it?

[192] S: No... we will now draw $|f(|x|)|$.

[193] H: Do you know how? Well, the next step is not too clear to me!

[194] I: OK, just to make your job a bit easier, let's rename $f(|x|)$ as $g(x)$. So what you need to find turns into...

[195] S: ... $|g(x)|$.

[196] H: Aha! I can see it now...

[197] I: What is it?

[198] H: That means we will draw the absolute value of this graph... I mean we need to take the absolute value of this graph... Oh, it is so clear now. Do you understand, S?

[199] S: Of course, but renaming the expression helped me see it clearly now.

[200] I: OK, let's think about it now, how can we apply absolute value to this graph?

[201] S: $|g(x)|$ never takes negative values... I mean it never passes under the x-axis.

[202] H: We will be taking the reflection of the rays [the line segments in Figure 2-D] under the x-axis.

H&S constructed the graph of $|f(|x|)|$ in two steps: first drawing the graph of $f(|x|)$ (Figure 2-D) and then applying their earlier method of drawing the graph of $|f(x)|$ to the graph of $f(|x|)$. We call this the two-step method. H&S saw that this gave the same graph as that obtained by substitution.

H&S proceeded to draw the graph of $|f(|x|)|$ for the third question using their two-step method and did so by first drawing the graph of $g(x) = |x| + 3$ and then drawing the graph of $|g(x)|$ (lines 213-223, not shown). Again they realised that they obtained the same graph of $|f(|x|)|$ as they had previously obtained by substituting. They also realised that the graph of $|f(|x|)|$ in the third question was the same graph as $f(|x|)$. They then discussed why, in this particular question, the graphs of $f(|x|)$ and $|f(|x|)|$ were the same:

[224] H: Yes, that means we don't take any symmetry after drawing the graph of $f(|x|)$...because the graph of $f(|x|)$ is already above the x-axis... That means, for the third question, the graph of $|f(|x|)|$ is the same graph as $g(x)$ [$=f(|x|)$]... I mean, this is V-shaped.

[225] S: Look, it must be so [V-shaped]... because for the third question, look at this equation [$|(|x| + 3)|$], even if the absolute value at the outside of the whole expression is removed, we still obtain the same values of y... I mean every value of y is positive for $f(|x|)$ and so the absolute value sign outside the whole expression doesn't make any difference... So these two graphs should be the same anyway...

H&S were satisfied that this graph was V-shaped, which was once a hindrance to their progression (see Episode 1) and concluded that their method worked — they had constructed a new method.

[244] H: First, when drawing $f(|x|)$, part of $f(x)$ at the positive [values of] x remains unchanged... Umm, then this part is taken symmetrically in the y-axis and, err, and also part of f(x) at the negative [values of] x is cancelled. After that, we apply absolute value to this graph [$f(|x|)$] and for this... umm... negative values of y are taken symmetrically in the x-axis and thus we obtain the graph [of $|f(|x|)|$].

Discussion

We now attend to the four questions we posed in the introduction:

How are abstractions mediated and what kind of mediation are we talking about?

Dreyfus, Hershkowitz, and Schwarz (2001) argue that vertical reorganisation of previously constructed mathematics involves the use of artefacts available to a student(s). Artefacts, to them, include material objects and tools but also language and procedures; in the RBC model, the term includes everything which potentially mediates students' construction of new knowledge structures.

Construction of new mathematical structures takes place through the nested epistemic actions of recognising, building-with, and constructing. We hold that epistemic actions, like all cultural actions, are undertaken through mediation which applies equally to (knowledge) artefacts and people (Cole, 1996). In this section, we consider artefact and human mediation separately although they are intertwined in student actions.

We can see in the protocol excerpts the mediation of epistemic actions through the knowledge artefacts at students' disposal. For example, in Episode 1, H&S recognised symmetries and reflections in particular graphs of $|f(|x|)|$, and these recognitions were mediated by the graphs themselves and features of Cartesian grids. For example, they talked about rays, lines, and quadrants: "first quadrant" [139], "y-axis" [136], and "x-axis" [137]. H&S's analogy of computational order and parentheses (Episode 2) regarding the graphs of $|f(|x|)|$ provides a nice illustration of building-with actions. The students recognised the computational priority of parentheses [170], gave examples to illustrate the order [172], and related earlier constructions of $f(|x|)$ and $|f(x)|$ to $|f(|x|)|$. These knowledge artefacts mediated their building-with actions. H&S's constructing actions, especially evident in Episodes 2-4, were similarly mediated: H&S assembled knowledge artefacts (computational priority, order of parentheses, features of $f(|x|)$ and $|f(x)|$, symmetry, and properties of euclidian geometry) and, by establishing new connections amongst them, merged them into a new structure: the two-step method for $|f(|x|)|$.

We now turn to human mediation, an area barely touched upon by Hershkowitz, Schwarz, and Dreyfus (2001). H&S's work was mediated by the interviewer's interventions: The interviewer acted as a knowledge artefact which the students made essential use of in their construction of the two-step method. We do not put value judgements (good or bad) on the quality of this interviewer mediation but merely claim that many of the manifested epistemic actions were undertaken as a result of the interviewer interventions. Take, for example, H's act of recognition that "$|f(|x|)|$ is a combination of $f(|x|)$ and $|f(x)|$" [170]. This recognition was mediated by the interviewer's prompts in [165] and [168], after which she exclaimed "It makes sense!" Here, we do not see H's utterance as a simple repetition of the interviewer's utterance because she used this recognition in connecting the expression of $|f(|x|)|$ with computational precedence

(building-with) and even gave an example [172]. Thus both the act of recognising and the resulting building-with were mediated by the interviewer's intervention.

But what role did the interviewer interventions play in H&S's development? One way to view the interviewer mediation might be in terms of facilitation: The interventions facilitated H&S's mathematical actions. Although we take this to be true, we believe that the interviewer's mediation led to significant changes in H&S's performance which seem unlikely to have occurred without intervention. The interventions did not simply facilitate H&S's progress — they transformed their ways of seeing, talking, and acting. To better appreciate this argument, we make comparative comments on H&S's work in Episode 1 and Episodes 2-4. In Episode 1, H&S's work essentially involved recognising and building-with actions and interviewer interventions were few and unimportant. In the later episodes, however, the interviewer was involved in H&S's work and his interventions played a crucial role in H&S's construction.

In Episode 1, H&S focused on two graphs of $|f(|x|)|$ with different shapes. They were clearly confused by the shapes of the graphs and even stated their intention to give up. However, from Episode 2, a transformation is apparent in the students' ways of seeing: They were looking at the same expression, $|f(|x|)|$, but seeing in it their earlier constructions of $|f(x)|$ and $f(|x|)$ and seeing a relation to the idea of precedence of operations. H&S's ways of seeing, aided by further interviewer interventions, eventually led them to see a reason why graphs of $|f(|x|)|$ did not go below the x-axis and to explain the different shapes of the graphs.

Throughout Episodes 2-4, there was a dramatic change in the way that H&S talked about the expression of $|f(|x|)|$ and the graphs. In Episode 1, H&S talked about the similarities and differences between the sketched graphs of $|f(|x|)|$ and focussed on visual characteristics such as line segments and lines of symmetry. However, in the subsequent episodes, they were talking about certain structures which are not immediately available to the senses such as the analogy to computational order in the expression $|f(|x|)|$ (Episode 2), possible relations between the structures of $|f(x)|$, $f(|x|)$, and $|f(|x|)|$ (Episode 3), and successive applications of $f(|x|)$ and $|f(x)|$ to a given linear graph to obtain the target graph (Episode 4).

Finally, these transformations were also reflected in H&S's actions. In Episode 1, H&S were mainly acting at recognising and building-with levels as they merely reported similarities and differences and attempted to explain the different graphs on this basis. However, in Episodes 2-4, H&S's actions were transformed into constructing actions as they now combined and assembled knowledge artefacts, merging the graphs of $f(|x|)$ and $|f(x)|$ into a single graph and linking them to sketch the target graphs.

A critical question is, under which conditions do such transformations come about through human mediation? We cannot provide a fine-grained answer to this, but our analyses convince us of the importance of intersubjectivity, the establishment of common ground amongst the participants, and, more specifically, the extent to which participants in a communicative situation share a perspective (Matusov, 2001; Wertsch, 1998). Intersubjectivity, for example, can be seen when the students appreciated the interviewer's interventions and were

able to draw on what had been brought to their attention (see [165-170] and [194-199]). In this sense, the participants shared a common perspective and managed to establish common ground. However, further research is needed to clarify how human mediation achieves these transformations.

What is the Role of Tutor Interventions in Guiding Students?

We use the word *tutor* as opposed to teacher because our data is based on one tutor and two students — a teacher typically has many more than two students and this brings a host of problems that our data cannot address. We consider three functions of tutor interventions which we believe are important in explaining the transformations in H&S's development: reducing uncertainty, directing student attention, and setting sub-goals.

Reducing students' uncertainty appears to be important for their development. Generally speaking, in the course of construction of a new structure, uncertainty seems to be inevitable because during construction one needs not only to recognise and use but also to reorganise available knowledge structures, relate them to the new situation, and forge new connections amongst them. Furthermore, all these actions need to be carried out in an unfamiliar situation; for construction is the process through which students become familiar with the new structure, which presupposes students' unfamiliarity with the to-be-constructed structure prior to construction. Given that students have no clear picture of the construction to be formed (for otherwise, it would have been formed already), they necessarily confront uncertainty. Wood (1991) describes the relationship between unfamiliarity and uncertainty:

> When we find ourselves needing to act in a very unfamiliar situation, uncertainty is high and our capacity to attend to and remember objects, features and events within the situation is limited... Children... are potentially confronted with more uncertainty than the more mature... Without help in organising their attention and activity, children may be overwhelmed by uncertainty. The more knowledgeable can assist them in organising their activities, by reducing uncertainty, breaking down a complex task into more manageable steps or stages. (pp. 105-106)

H&S's uncertainty about the appropriateness of their proposals, elaborations, and explanations appeared and reappeared throughout their work. For instance, following their suggested analogy to computational precedence [170-174], H&S were uncertain as to the appropriateness of this analogy and how to "determine the parenthesis" in the expression of $|f(|x|)|$. The interviewer made an encouraging intervention and gave a short explanation. Later, H&S expressed their uncertainty as to the aptness of approaching the graphs of $|f(|x|)|$ through the successive application of $f(|x|)$ and $|f(x)|$ [185]. The students here needed to forge new connections amongst the available structures of $f(|x|)$ and $|f(x)|$ and the idea of computational priority. Surely the students' uncertainty stemmed from the fact that they were confronting such an issue for the first time and were thus acting in an unfamiliar situation. They also lacked a vision of the target

construction. The interviewer then reassured H&S, specified a target, and helped H&S to continue and eventually construct the two-step method.

As Wood hinted, one way to help students deal with uncertainty is to organise their attention — which brings us to the second function of interventions: to direct the students' attention. The management of attention in collaborative learning environments is considered important to achieve new learning (Barron, 2003; van Oers, 2001). Mason (1989) attributes a focal role in doing and learning mathematics to shifts of attention. As Mason and Spence (1999, p.151) write:

> Coming to know is essentially a matter of shifts in the structure of attention, in what is attended to, in what is stressed and what consequently ignored with what connections... Knowing is not a simple matter of accumulation. It is rather a state of awareness, of preparedness to see in the moment. That is why it is so vital for students to have the opportunity to be *in the presence of someone who is aware of the awarenesses that constitute their mathematical "seeing."* (emphasis added)

In the case of a construction, this "seeing" contributes to successful use of available structures (building-with) when required within the situation. However, as Mason and Spence (1999, p.135) propose, the use of available knowledge "depends on what one is aware of" because "no-one can act if they are unaware of a possibility to act; no-one can act unless they have an act to perform." This suggests that if students are not aware of the importance and necessity of the knowledge artefacts available to them for a new construction, they are unlikely to draw upon them as they (or their attention) are blocked. This was the case when H&S failed to see the connection between their knowledge of $|f(x)|$ and $f(|x|)$ and the expected construction of $|f(|x|)|$. It was here that the interviewer's efforts to direct H&S's attention became important. The interviewer first brought $|f(x)|$ and $f(|x|)$, and what they knew about them, to H&S's attention. He then drew their attention to the expression of $|f(|x|)|$ and suggested viewing this as a combination of $|f(x)|$ and $f(|x|)$ [165]. Only then did he invite the students to work out an idea as to how to use $|f(x)|$ and $f(|x|)$ in connection with $|f(|x|)|$.

The interviewer's effort to direct H&S's attention continued during their work, especially at the moments of uncertainty. For instance, in Episode 4, when applying their strategy to draw the graph of $|f(|x|)|$ in two steps, H&S first drew $f(|x|)$; but their uncertainty reappeared as to how to proceed from this graph to the target graph. The interviewer intervened and labelled $f(|x|)$ as $g(x)$, implying that drawing the graph of $|f(|x|)|$ was the same as drawing the graph of $|g(x)|$. The interviewer's role, here and elsewhere, could be described, in Bruner's (1985, p. 24) terms, as serving as the students' "vicarious form of consciousness" due to the fact that he knew in what direction to guide H&S.

In both reducing H&S's uncertainty and directing their attention, the interviewer set sub-goals. The most common ways in which sub-goals were set were direct requests — for example, "How can we use our knowledge to obtain this graph?" [168] and "Let's draw the graph" [187]. We can see H&S working to fulfil these goals, which then moved them closer to the construction of the two-step method. These sub-goals were not predetermined but emerged during

interaction. They were dialectically shaped by the interviewer's understanding of the students' development at certain stages in the activity and the students' understanding of the interventions in the context of a particular task (cf. Saxe, 1991). However, the emergence of these sub-goals in relation to H&S's knowledge construction in socially rich environments is a complex matter. We do not have space for a detailed consideration of this complexity and refer the reader to Ozmantar (2004, 2005) and Ozmantar and Monaghan (2006). Yet we wish to note here that this complexity stems mainly from the differences in the perspectives and understandings of the participants, not only regarding the mathematical content — the interviewer has a clear vision of the target construction and the possible ways to achieve this but the students do not — but also regarding the nature and the reconciliation of emergent goals which differ with the agent. We believe that this issue is an important one and that the emergent goals of the participants in the course of knowledge construction in socially rich environments deserve further investigation.

What are the Implications of a Dialectic View of Abstraction on Student Development?

As we noted in the introduction, the discovery of the essence is central not only to dialectical views but also to the empiricist understanding of abstraction. Empiricists posit the recognition of commonalties among a large number of particular instances. Davydov's dialectical account, however, starts with an analysis of the initial stages of abstraction and proceeds to synthesis. Analysis is used to connect the observable features of reality through empirical thought (e.g., observing similarities and differences, designating the contradictions) but synthesis requires theoretical thought to establish real interconnections and to find a single source, a universal base, for a variety of phenomena.

In H&S's protocol excerpts, one can see these students initially engaging in an analysis process through which they designate contradictions between the graphs of $|f(|x|)|$ (W- and V-shaped graphs, presence or absence of certain rays and line segments) by observing similarities and differences between the graphs. However, these observations were not sufficient for them to come up with a structure of $|f(|x|)|$ as their consideration did not go beyond the analysis stages. By contrast, we can see H&S engaging in synthesis in Episodes 2-4 — but only after the involvement of the interviewer. For instance, H&S formed an analogy between the expression of $|f(|x|)|$ and computational order which is not directly available to the senses (i.e., it was not an observable communality).

What was happening during this synthesis? The students were going through cycles of changes and reproducing "the universal forms of things, their measures and their laws" (Davydov, 1972/1990, p. 249). In this regard, Davydov writes:

> Within the evolving natural whole, all things are constantly changing, passing into other things, vanishing. But each thing, according to dialectics, does not merely change or disappear — it passes into its own other, which, within some broader interaction of things, proceeds as a necessary consequence of the being

> of the thing that has vanished, retaining everything positive from it within the limits of all nature this is also a universal connection. (p. 253)

In relation to our study, Davydov's statement implies that construction takes place through cycles of constant changes involving increasing clarification and progressive evolution of the initial form of a to-be-constructed structure. In the course of this evolution, new understandings emerge whilst others (e.g., preliminary understandings) vanish, though "retaining everything positive" from them. To exemplify this point, we return to Episode 2. After the interviewer brought the structures of $|f(x)|$ and $f(|x|)$ to the students' attention, they recognised a form of computational precedence in the expression $|f(|x|)|$. They stated old knowledge about computational order with a particular example [172] but for a new purpose. Their perception of the expression of $|f(|x|)|$ changed, "passed into" a new form that they "can sense" but were "unable to clarify" [174]. In Episode 3, following further interviewer interventions, H&S's initial suggestion of viewing $|f(|x|)|$ from the perspective of computational order became a firm strategy for applying the structures of $|f(x)|$ and $f(|x|)$ to a linear graph in order to obtain the target graph.

Throughout H&S's interactions, some things vanished and passed into other things. For example, in [179] H suggested considering $f(|x|)$ as the "smaller parenthesis." This was playing with a mathematical analogy (but an analogy which has epistemic value) and represents an expression of a preliminary understanding. This analogical expression later disappeared but the knowledge it generated lived on. The result can be seen in Episode 4, where consideration of $f(|x|)$ as the smaller parenthesis had given rise to the idea of transforming $f(x)$ to $f(|x|)$ as a first step.

Another aspect of the dialectical view is that particular instances become more sensible after the construction. To appreciate this point, recall that H&S were perplexed by the W and V shapes of the two graphs obtained for Question 1 and 3 [142]. However, following the construction of the two-step method, H&S comprehended why the graph of $|f(|x|)|$ in Question 3 should be V-shaped. In Davydov's terms, H&S achieved a grasp and rational expression "not only of the existence of certain things and their properties, but also of their possibility, as such, with a subsequent determination of the conditions of their manifestation in a certain form" (1972/1990, p. 278). The students understood that the manifestation of the graph of $|f(|x|)|$ in an unexpected V-shaped form "must be so" and worked out the conditions under which this would happen [224-225].

These observations point out difficulties with views that attribute epistemological primacy to particular instances over abstractions. It is important to note that H&S, in constructing the two-step method, recognised and used existing knowledge. Hence it was recognition and use of previously abstracted knowledge, not to-be-abstracted knowledge, that led H&S to form new abstractions. Such a view does not admit persistent interests or causal beliefs (see earlier discussion of Bartlett and of Keil). We do not deny that causal beliefs exist. But they cannot explain construction or abstraction, since H&S initially believed in the existence of certain causes in explaining the shapes of the graphs but these

did not explain the construction and only led to it by "passing into" other forms. What H&S were lacking in, in Episode 1, was the assistance of an agent who, in a given situation, had awareness of what to attend to and of what to ignore (Mason & Spence, 1999) to take the students into a process of synthesis.

What Kind of Things are Abstracted?

One reason, as mentioned, for our interest in the RBC model of abstraction, over other dialectical accounts, is that the identification of epistemic actions allows one to be precise about the microgenetic development during the abstraction process. Process aspects of an abstraction are important to researchers but so is the product of abstraction. Hershkowitz, Schwarz, and Dreyfus (2001) use Davydov's term *structure* for the product of an abstraction. The term structure is further used (Dreyfus & Tsamir, 2004) as a generic term for abstracted methods, strategies, and concepts. However, structures can be expanded to include, for example, mathematical terms and laws and, within a mathematical-theory, definitions (Clancey, 2001). So the question of interest is: What is a structure?

This question is not peculiar to the RBC model, and related questions are asked in related domains. With regard to internalisation, Wood and Wood (1996, p. 15) ask, "[Is it speech or] rules of action, in the service of goals, which become activated by symbol systems such as language and diagrams"? With regard to appropriation, a potential alternative to internalisation for those who have a problem with internalisation (Wertsch, 1998), Moschkovich (2004, p. 54) notes that "learners have been described as appropriating a broad spectrum of things ranging from information or skills, to meanings for words, to interpretations of a task, to ways of acting and thinking, or to discourses and social practices." In the framework of Chevallard's (1999) anthropological theory of didactics — which views mathematical practice in terms of task, technique, technology, and theory — structure might be considered as a technique or theoretical element.

Returning to H&S's work reported here, the structure they constructed was their two-step method. In the course of this construction, however, H&S developed a language to describe their method [244]. So should the term structure also involve some sort of language development? H&S's construction brought about transformations in their ways of seeing and acting — are these also part of the structures? Stating what is constructed (or abstracted) in a general manner is not a simple, and may not be a fruitful, endeavour precisely because the construction of a structure involves multifaceted development on the part of students in their ways of acting, seeing, talking, thinking, using, recognising and so on. Considering these complexities, we do not have a problem with using the term structure in a generic way to describe what the students acquire or develop, but we believe that it is important for studies concerned with abstraction to describe in detail precisely what structures students construct.

Conclusion

We return to the issue we raised at the beginning of this paper, the dialectical nature of abstraction. We cannot view H&S's abstraction, as some empiricists might, as a decontextualised ascent from the concrete to the abstract derived from the recognition of commonalities across a set of particular instances because context was crucial: the given task in a sequence of related tasks, how these tasks were structured and what H&S had to attend to, H&S's mathematical histories prior to and within the sequence of tasks, and their interactions with each other and with the tutor. Furthermore, H&S's abstraction did not arise simply from recognition of commonalities but, rather, from creating meaning by relating new observations to what they already knew. This, to us, is an ascent to the concrete — a process of making meaning by establishing interconnections amongst elements of the whole — and this is dialectics.

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